of consistent boundary conditions for general three-dimensional rotordynamic configurations and time-discretization truncation error.

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Boundary Integral Equations for Notch Problems in Plane Thermoelasticity

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Introduction

THE magnification of stresses at geometric discontinuities is of great importance in engineering design. In particular, local stresses may be highly enhanced in notched materials arising from abrupt changes of shape. This would result in a substantial decrease of the load-bearing capacity of structural members. Various methods of calculating stress concentration factors have been developed for two-dimensional elasticity problems. Using a series expansion method, Ling[1] solved elastic problems with different types of notches. Bowie and Freese[2] analyzed the notch problem by using complex variable theory in conjunction with the conformal mapping method. Nisitani[3] used the method of body force (or Green's function) to solve the notch problem in a semi-infinite plate or in a strip.

An alternative method for solving notch problems may be formulated in terms of a system of boundary integral equations. This method has clear advantages in solving the problem by applying a numerical treatment. In the derivation of boundary integral equations, the selection of the auxiliary function determines whether the kernels have weak or strong singularities. The kernel with Cauchy-type singularity has been widely used to solve many crack problems.[4] On the other hand, the integral equation with a logarithmic kernel has been proved to easily perform the numerical computation by Cheung and Chen.[5] This method with weak singularity has been used to solve some crack problems associated with an elastic half-plane medium,[6] two bonded thermoelastic half-plane media,[7] and thermoelastic circular inclusion perfectly bonded in an infinite matrix.[8] Based on the earlier derivation, Chen and Cheung[9] recently reformulated a new boundary integral equation to deal with the notch problem in plane elasticity. In this Note, we aim to further extend the aforementioned method to solve notch problems in plane thermoelasticity. In the derivation of singular integral equations, instead of using the components of heat flux and the components of stress, the resultant heat flow $\boldsymbol{Q}$ and the resultant force $-Y + iX$ are used to formulate the boundary conditions along the notch surface. This would result in singular integral equations with a logarithmic kernel instead of a Cauchy-type kernel. Three different types of notches in an infinite medium under a remote uniform heat flow are considered as our examples to illustrate the use of the approach. Some available exact solutions are provided to compare with the calculated numerical results to demonstrate the accuracy of the study.

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Formulation of Integral Equation: Thermal Field

For the two-dimensional steady-state heat conduction problem, the temperature function, which satisfies the Laplace equation, can be expressed in terms of a single analytic function $\theta(\boldsymbol{z})$. With this function, both the temperature T and the resultant heat flow $\boldsymbol{Q}$ are written as

$$T = \mathrm{Re}[\theta(\boldsymbol{z})] \tag{1}$$

$$\boldsymbol{Q} = \int (\boldsymbol{q}_x \, \mathrm{d}y - \boldsymbol{q}_y \, \mathrm{d}x) = -k\,\mathrm{Im}[\theta(\boldsymbol{z})] \tag{2}$$

where Re and Im denote the real and imaginary parts of the bracketed expression, respectively. The quantities $\boldsymbol{q}_x$ and $\boldsymbol{q}_y$ in Eq. (2) are the components of heat flux in the x and y directions, respectively, and k is the heat conductivity. Consider a remote uniform heat flux approached from the negative x axis is obstructed by the presence of an insulated notch or hole in an infinite medium. The current problem can be treated as a sum of the corresponding infinite medium problem without notches and a corrective term. The solution associated with the former problem can be easily expressed as

$$\boldsymbol{Q}_0(\boldsymbol{z}) = \boldsymbol{q}\,\mathrm{Im}[\boldsymbol{z}] \tag{3}$$

with $\boldsymbol{q}$ being the strength of heat flux applied at infinity.

On the other hand, a corrective solution associated with an infinite medium with a single notch can be obtained by assuming a continuous distribution of dislocations with the density $b_0(s)$ placed along a given contour L as

$$\theta(\boldsymbol{z}) = -\frac{i}{2\pi}\int_L \log(\boldsymbol{z} - t)\, b_0(s)\, \mathrm{d}s \tag{4}$$

The resultant heat flow across the notch surface can be obtained by substituting Eq. (4) into Eq. (2) as

$$\boldsymbol{Q}(\boldsymbol{z}) = \frac{k}{4\pi}\int_L [\log(\boldsymbol{z} - t) + \log(\bar{\boldsymbol{z}} - \bar{t})]\, \mathrm{d}s + c_0, \qquad \boldsymbol{z} \in L \tag{5}$$

where a bar will be used to indicate a conjugate complex quantity and c_0 is a constant to be determined.

Based on the superposition principle, the boundary integral equation for an infinite medium containing an insulated notch is then established as follows:

$$\frac{k}{2\pi}\int_L \log(|\boldsymbol{z} - t|)\, b_0(s)\, \mathrm{d}s + c_0 = -\boldsymbol{q}\,\mathrm{Im}[\boldsymbol{z}], \qquad \boldsymbol{z} \in L \tag{6}$$

In addition, the single-valued condition of the temperature must be satisfied, i.e.,

$$\int_L b_0(s)\, \mathrm{d}s = \boldsymbol{0} \tag{7}$$

Equation (6) together with Eq. (7) constitutes a boundary integral equation for solving the unknown function $b_0(s)$. Once the function $b_0(s)$ is determined, the temperature function $\theta(\boldsymbol{z})$ in Eq. (4) will be obtained accordingly.

Formulation of Integral Equation: Thermal Stress Field

For a two-dimensional thermoelastic problem, the components of the displacement and the traction force can be expressed in terms of two analytic functions, $\phi(z)$ and $\psi(z)$, and a temperature function $\theta(z)$ as[10]

$$2G(u + iv) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} + 2G\beta\int \theta(z)\,\mathrm{d}z \qquad (8)$$

$$-Y + iX = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \qquad (9)$$

where G is the shear modulus; $\kappa = (3 - \nu)/(1 + \nu)$ and $\beta = \alpha$ for plane stress; and $\kappa = 3 - 4\nu$ and $\beta = (1 + \nu)\alpha$ for plane strain, with ν Poisson's ratio and α the thermal expansion coefficient. The stress functions associated with an infinite medium containing a single notch can be obtained by assuming a continuous distribution of edge dislocations with densities $b_1(s)$ and $b_2(s)$ placed along a given contour L as

$$\phi(z) = \frac{iG}{\pi(1+\kappa)}\int_L [b_1(s) + ib_2(s)]\log(z - t)\,\mathrm{d}s \qquad (10)$$

$$\psi(z) = \frac{-iG}{\pi(1+\kappa)}\int_L [b_1(s) - ib_2(s)]\log(z - t)\,\mathrm{d}s - \frac{iG}{\pi(1+\kappa)}\int_L \frac{[b_1(s) + ib_2(s)]\bar{t}}{z - t}\,\mathrm{d}s \qquad (11)$$

Substituting Eqs. (10) and (11) into Eq. (9) yields

$$-Y + iX = \int_L K_1(t, \bar{t}, z, \bar{z})[b_1(s) + ib_2(s)]\,\mathrm{d}s + \int_L K_2(t, \bar{t}, z, \bar{z})[b_1(s) - ib_2(s)]\,\mathrm{d}s + c_1 + ic_2, \qquad z \in L \qquad (12)$$

where

$$K_1(t, \bar{t}, z, \bar{z}) = \frac{2iG}{\pi(1+\kappa)}\log|z - t| \qquad (13)$$

$$K_2(t, \bar{t}, z, \bar{z}) = \frac{iG}{\pi(1+\kappa)}\frac{t - z}{\bar{z} - \bar{t}} \qquad (14)$$

For the traction-free condition along the notch surface, we now have the boundary integral equation

$$\int_L K_1(t, \bar{t}, z, \bar{z})[b_1(s) + ib_2(s)]\,\mathrm{d}s + \int_L K_2(t, \bar{t}, z, \bar{z})[b_1(s) - ib_2(s)]\,\mathrm{d}s + c_1 + ic_2 = 0 \qquad (15)$$

Furthermore, the single-valued condition of the displacement must be satisfied, i.e.,

$$\int_L [b_1(s) + ib_2(s)]\,\mathrm{d}s - \int_L \beta\left[\int b_0(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s = 0 \qquad (16)$$

Equation (15) together with Eq. (16) constitutes a boundary integral equation for solving the unknown functions $b_1(s)$ and $b_2(s)$. Once these two functions are determined, the stress functions $\phi(z)$ and $\psi(z)$ in Eqs. (10) and (11), respectively, will be obtained accordingly.

Numerical Results and Discussion

The dislocation functions $b_0(s)$ in Eq. (6), and $b_1(s)$ and $b_2(s)$ in Eq. (15) together with the subsidiary conditions [Eqs. (7) and (16)] will be solved numerically using the appropriate interpolation formulas. For performing the numerical calculation, the contour L is replaced by a polygon of N line segments. The interpolation formulas for line segments in local coordinates $s_j (1 \le j \le N)$ are taken as[9]

$$b_i(s_j) = b_{i,j}\frac{d_j - s_j}{2d_j} + b_{i,j+1}\frac{d_j + s_j}{2d_j} \qquad (i = 0, 1, 2) \qquad (17)$$

where $d_j (1 \le j \le N)$ are the half-length for each line segment and $b_{i,j} (0 \le j \le N)$ are the unknown coefficients to be determined. If the preceding formulas are used, the boundary integral equation [Eq. (6)] together with the subsidiary condition [Eq. (7)] can be carried out to yield $N + 2$ algebraic equations for solving $N + 2$ unknown coefficients ($b_{0,0}, b_{0,1}, b_{0,2}, \ldots, b_{0,N}, c_0$). Similarly, the boundary integral equation [Eq. (15)] together with the subsidiary condition [Eq. (16)] can be arranged to yield $2N + 4$ algebraic equations for solving $2N + 4$ unknown constants ($b_{1,0}, b_{1,1}, \ldots, b_{1,N}, b_{2,0}, b_{2,1}, \ldots, b_{2,N}, c_1, c_2$). Once the stress functions are determined, the tangential stresses or hoop stresses along the notch surface may be evaluated by

$$\sigma_t = 4\,\mathrm{Re}\{\phi'(z)\}, \qquad z \in L \qquad (18)$$

Circular Hole

As our first example, an insulated circular hole in an infinite medium under a remote uniform heat flow is considered. For performing the numerical technique, the contour of the circular hole is replaced by a polygon of N line elements discreted with a number of N points expressed by

$$x_i = a\cos\left[\frac{2(i-1)\pi}{N}\right], \qquad y_i = a\sin\left[\frac{2(i-1)\pi}{N}\right] \qquad (i = 1, 2, \ldots, N) \qquad (19)$$

The calculated hoop stresses and the corresponding exact results are displayed in Fig. 1. It can be seen that the calculated numerical results agree very well with the corresponding exact solutions with the number of line segments $N = 48$.

Elliptic Hole

As our second example, we consider an insulated elliptic hole with the semi-axes a and $b = a/2$ under a remote uniform heat flow. Similar to the preceding approach, some discreted points along the elliptic hole are expressed by

$$x_i = a\cos\left[\frac{2(i-1)\pi}{N}\right], \qquad y_i = b\sin\left[\frac{2(i-1)\pi}{N}\right] \qquad (i = 1, 2, \ldots, N) \qquad (20)$$

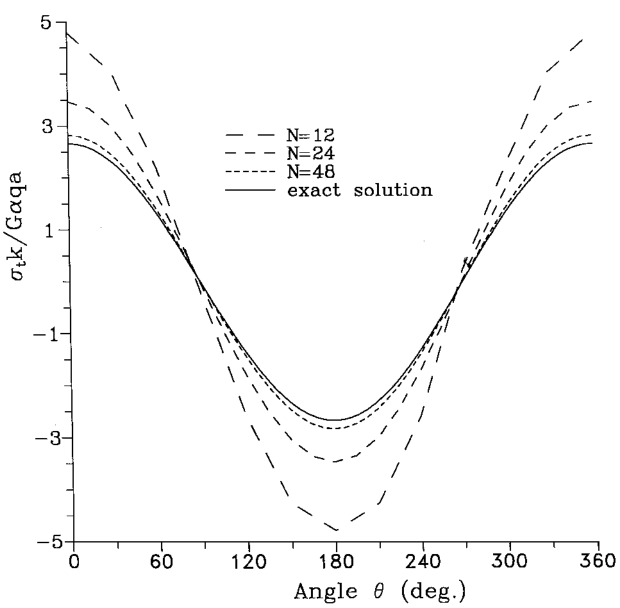

Fig. 1 Comparisons between the calculated and exact values of hoop stresses of the circular hole.

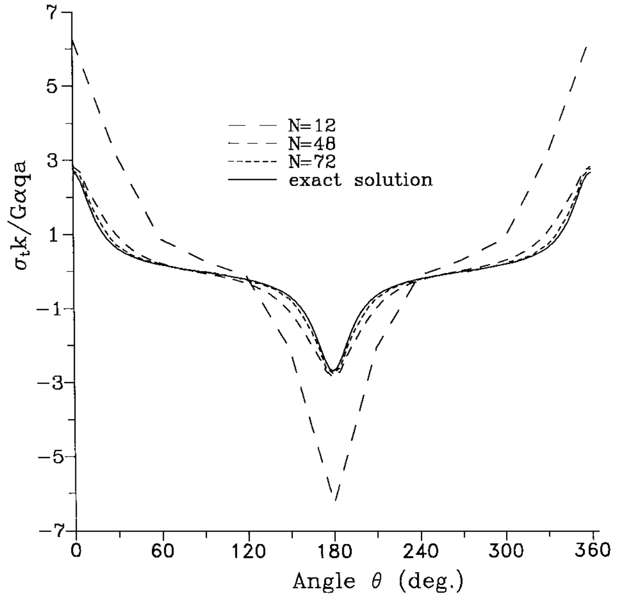

Fig. 2 Comparisons between the calculated and exact values of hoop stresses of the elliptic hole.

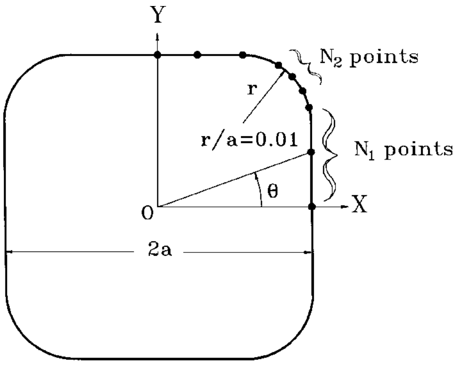

Fig. 3a Insulated square hole embedded into an infinite matrix.

The calculated hoop stresses and the corresponding exact results are displayed in Fig. 2. It shows that the error between the numerical results and exact solutions is within 2% with the number of line segments $N = 72$.

Square Hole

As our third example, the notch problem of square hole with round conners is considered and shown in Fig. 3a. In the following numerical analysis, the choice of N_1 points is selected along the straight portions, whereas the choice of N_2 is selected along the round conners. The exact results and numerical hoop stresses of two cases with $N_1 = N_2 = 2$ and $N_1 = N_2 = 4$ are displayed in Fig. 3b. Good accuracy is also observed for the square hole problem with $N_1 = N_2 = 4$.

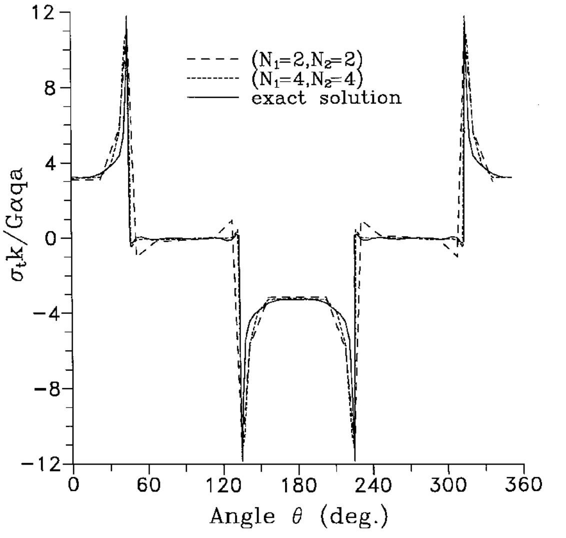

Fig. 3b Comparisons between the calculated and exact values of hoop stresses of the square hole.

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